Table 1 $[D]-[I]$ matrices for different mass profiles

Uniform beam					
2.04976	0.00000	−0.58745	0.00000	−0.09087	0.00000
0.00000	5.39740	0.00000	−1.86112	0.00000	−0.36582
−0.58745	0.00000	10.32461	0.00000	−3.75429	0.00000
0.00000	−1.86112	0.00000	16.90500	0.00000	−6.19466
−0.09087	0.00000	−3.75429	0.00000	25.12963	0.00000
0.00000	−0.36582	0.00000	−6.19466	0.00000	34.99923
Tapered beam					
1.33718	0.00000	−0.11555	0.00000	−0.16446	0.00000
0.00000	3.44538	0.00000	−0.63192	0.00000	−0.35044
−0.11555	0.00000	6.14599	0.00000	−1.14513	0.00000
0.00000	−0.63192	0.00000	9.83703	0.00000	−2.02919
−0.16446	0.00000	−1.14513	0.00000	14.70739	0.00000
0.00000	−0.35044	0.00000	−2.02919	0.00000	20.35255

Table 2 Eigenvalues of $[D]-[I]$ for different orders of approximation

Order	Eigenvalues					
	Uniform beam					
1	2.0498					
2	2.0498	5.3974				
3	2.0083	5.3974	10.3661			
4	2.0083	5.1039	10.3661	17.1985		
5	2.0024	5.1039	9.4743	17.1985	26.0273	
6	2.0024	5.0386	9.4743	15.3450	26.0273	36.9181
	Tapered beam					
1	1.3372					
2	1.3372	3.4454				
3	1.3344	3.4454	6.1488			
4	1.3344	3.3835	6.1488	9.8989		
5	1.3316	3.3835	5.9994	9.8989	14.8595	
6	1.3316	3.3653	5.9994	9.5361	14.8595	20.7336

and the integral of this matrix is also positive definite. Therefore, we conclude that, for a beam of arbitrary mass and stiffness distributions undergoing planar motion but not subject to external axial loading, centrifugal stiffening effects always dominate geometrical softening effects.

Numerical Example

As an illustrative example, we will consider two symmetric free-free beams of length of $2L = 144$ length units: a uniform beam and a linearly tapered beam. Letting x be measured from the center $(-72 \leq x \leq 72)$, the respective mass profiles of these beams are $m(x) = M$ and $m(x) = M[0.2 + 1.6(L - |x|)/L]$, where the total mass of each beam is 0.144 units ($M = 0.001$). The stiffness distribution is analogous: M is replaced by $EI = 10{,}000$ in the preceding distributions. The principal mass moments of inertia of the beams about the axes of rotation are 249 and 241, respectively.

Table 1 shows the matrix $[D]-[I]$ for each of the cases, while Table 2 shows the eigenvalues of $[D]-[I]$ for different orders of system approximation up to sixth order. The fact that these are all positive values indicates that the matrices $[D]-[I]$ are positive for all orders of approximation. Note that the eigenvalues are roughly in proportion to the inertias. Since we would expect centrifugal effects to be greater for a beam with mass concentrated away from the center of mass, this result is quite reasonable.

Conclusions

It is shown that, for a rotating pinned-free or free-free beam confined to planar motion and subject to no external axial loading, the stiffening effect always dominates the softening. Although the proof may be valuable in itself, it is meant more as a template for further results. In the equations describing fully developed three-dimensional flexible and rigid body motion, stiffening is not guaranteed. This point will be addressed in a forthcoming paper.

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Linear Quadratic Regulator Approach to the Stabilization of Matched Uncertain Linear Systems

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Introduction

THE robust stabilization problem has received considerable attention in recent years. Nonlinear robust control laws that stabilize uncertain linear systems satisfying matching conditions were developed by Leitmann.[1] In addition, for matched uncertain linear systems, linear robust stabilizing control laws were developed by Thorp and Barmish,[2] Barmish et al.,[3] and Schmitendorf and Barmish.[4] However, the determination of these robust stabilizing control laws for matched uncertain systems involves complicated calculations. More recently, Jabbari and Schmitendorf[5] gave a noniterative method for design of linear robust stabilizing control laws based on a Lyapunov function approach. For mismatched uncertain linear systems, Schmitendorf[6] presented a scalar search procedure for determining a linear robust stabilizing controller. Moreover, Riccati equation approaches, which adjust a scalar to achieve the stabilization of systems with uncertainty parameters bounded by constraint sets, were derived by Petersen and Hollot[7] and Schmitendorf.[8]

This Note concentrates on the development of linear robust stabilizing control laws for matched uncertain linear systems

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via linear quadratic regulator theory and Lyapunov stability theory. These robust stabilizing controllers can be applied to matched uncertain linear systems with either norm- or entry-bounded uncertainty matrices. The Note is organized as follows. First, the matching conditions for uncertain linear systems to be stabilized are defined. It is shown that a class of dynamic systems, described by second-order monic vector differential equations, often satisfies the matching conditions. Next, linear robust stabilizing controllers for matched uncertain linear systems that contain either norm- or entry-bounded uncertainty matrices are developed. In addition, it is shown that linear robust stabilizing controllers for matched uncertain systems always exist. The results are summarized in the conclusion.

Definitions of Uncertain Linear Systems

Consider the uncertain

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ the control; A and B the nominal system matrix and input matrix with appropriate dimensions, respectively; and $\Delta A(\cdot)$ and $\Delta B(\cdot)$ the associated continuous-time uncertainty matrices with appropriate dimensions. We assume that the nominal system (A,B) is controllable. The system in Eq. (1) is said to be matched if there exist continuous-time matrix functions $G(\cdot) \in \mathbb{R}^{m\times n}$ and $H(\cdot) \in \mathbb{R}^{m\times m}$, such that

$$\Delta A(t) = BG(t), \qquad \Delta B(t) = BH(t), \qquad \|H(t)\| < 1 \quad (2)$$

for all t, where the matrix norm used in Eq. (2) is defined as the maximum singular value of the matrix [i.e., $\|M\| = \sigma_{\max}(M) = \lambda_{\max}^{1/2}(M^TM) = \lambda_{\max}^{1/2}(MM^T)$ for any real matrix]. Our objective is to develop a simple, flexible, noniterative method for finding a linear state-feedback control law $u(t) = Kx(t)$ so that this control law stabilizes the matched uncertain system in Eq. (1).

It is important to note that a dynamic system[9] that can be modeled by a second-order monic vector differential equation is often a matched system. This fact can be verified as follows. Consider the second-order regular vector differential equation

$$[M + \Delta M(t)]\ddot{q}(t) + [D + \Delta D(t)]\dot{q}(t)$$
$$+ [S + \Delta S(t)]q(t) = [F + \Delta F(t)]u(t) \quad (3)$$

where $q(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^m$ are partial state and input vectors, respectively; and M, D, S, and $F \in \mathbb{R}^{m\times m}$ are the nominal mass, damping, stiffness, and force matrices, respectively, and $\Delta M(\cdot)$, $\Delta D(\cdot)$, $\Delta S(\cdot)$, and $\Delta F(\cdot)$ the respective uncertainty matrices. Suppose that there is no uncertainty (negligible, if any) in the mass matrix, i.e., $\Delta M(t) = 0$, then the regular vector differential expression in Eq. (3) can be expressed as a monic vector differential equation as

$$\ddot{q}(t) + [\hat{D} + \Delta\hat{D}(t)]\dot{q}(t) + [\hat{S} + \Delta\hat{S}(t)]q(t)$$
$$= [\hat{F} + \Delta\hat{F}(t)]u(t) \quad (4)$$

where

$$\hat{D} \triangleq M^{-1}D, \ \Delta\hat{D}(t) \triangleq M^{-1}\Delta D(t), \ \hat{S} \triangleq M^{-1}S$$

$$\Delta\hat{S}(t) \triangleq M^{-1}\Delta S(t), \ \hat{F} \triangleq M^{-1}F, \ \Delta\hat{F}(t) \triangleq M^{-1}\Delta F(t)$$

if $\det(M) \neq 0$. The state-variable realization of the second-order monic vector differential expression in Eq. (4) in a block companion form can be represented as

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) \quad (5)$$

where

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -\hat{S} & -\hat{D} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \hat{F} \end{bmatrix}$$

$$\Delta A(t) = \begin{bmatrix} 0 & 0 \\ -\Delta\hat{S}(t) & -\Delta\hat{D}(t) \end{bmatrix} = BG(t)$$

$$\Delta B(t) = \begin{bmatrix} 0 \\ \Delta\hat{F}(t) \end{bmatrix} = BH(t)$$

with $G(t) = [-\hat{F}^{-1}\Delta\hat{S}(t), \ -\hat{F}^{-1}\Delta\hat{D}(t)]$ and $H(t) = \hat{F}^{-1}\Delta\hat{F}(t)$, if $\det(\hat{F}) \neq 0$. Obviously, the system in Eq. (5) satisfies the matching conditions in Eq. (2) provided that $\|H(t)\| < 1$.

Remark 1. In general, for the matched uncertain linear system in Eq. (1), the matrices $G(t)$ and $H(t)$ in Eq. (2) can be obtained from the given $\Delta A(t)$ and $\Delta B(t)$ using the technique based on singular-value decomposition.[9] ■

Guaranteed Robust Stabilizing Controllers for Matched Systems

Consider the following matched uncertain linear system:

$$\dot{x}(t) = [A + BG(t)]x(t) + [B + BH(t)]u(t) \quad (6)$$

Suppose that the only information about the uncertainty matrices in Eq. (6) is

$$\|G(t)\| \leq \alpha \qquad \text{and} \qquad \|H(t)\| \leq \beta < 1 \quad (7)$$

for all t. The following theorem guarantees that a robust stabilizing controller exists for the matched uncertain system in Eq. (6) having the constraints in Eq. (7).

Theorem 1. Consider the matched uncertain linear system in Eq. (6) with norm-bounded uncertainty matrices described in Eq. (7). Let $Q \in \mathbb{R}^{n\times n}$ be any given symmetric positive-definite weighting matrix and ϵ any selected positive scalar satisfying $\epsilon \ \in (0, (1-\beta)/\alpha)$. And let $P \in \mathbb{R}^{n\times n}$ be the symmetric positive-definite solution of the following Riccati equation:

$$A^TP + PA - (1 - \beta - \epsilon\alpha)\, PBB^TP + (\alpha/\epsilon)I + Q = 0 \quad (8)$$

Then, a robust stabilizing control law is given by $u(t) = Kx(t)$, where $K = -\gamma B^TP$ with $\infty > \gamma \geq 1/2$, and the closed-loop system matrix $A_c(t) = A + BG(t) + [B + BH(t)]K$ is asymptotically stable for all admissible uncertainty matrices $G(t)$ and $H(t)$ in Eq. (7).

Proof. Define

$$Q(t) \ \triangleq -A_c^T(t)P - PA_c(t)$$
$$= -A^TP - PA + 2\gamma PBB^TP - G^T(t)B^TP - PBG(t)$$
$$+ \gamma PB[H^T(t) + H(t)]B^TP$$

From Eq. (8), it follows that

$$Q(t) = PB\{(2\gamma - 1 + \beta)I + \gamma[H^T(t) + H(t)]\}B^TP$$
$$+ [\epsilon\alpha PBB^TP + (\alpha/\epsilon)I - G^T(t)B^TP - PBG(t)] + Q$$
$$\geq PB[(2\gamma - 1 + \beta)I - 2\beta\gamma I]B^TP$$
$$+ [(\epsilon/\alpha)PBG(t)G^T(t)B^TP + (\alpha/\epsilon)I - G^T(t)B^TP$$
$$- PBG(t)] + Q$$
$$= (2\gamma - 1)(1 - \beta)PBB^TP + [\sqrt{(\epsilon/\alpha)}PBG(t) - \sqrt{(\alpha/\epsilon)}I]$$
$$\times [\sqrt{(\epsilon/\alpha)}PBG(t) - \sqrt{(\alpha/\epsilon)}I]^T + Q$$

Hence, $Q(t) \geq Q > 0$ for $\infty > \gamma \geq ½$. Thus, based on the Lyapunov stability theory,[10] $A_c(t)$ is asymptotically stable for $\infty > \gamma \geq ½$. ■

Remark 2. The Riccati expression in Eq. (8) is constructed to account for the uncertain linear system in Eq. (6) with the uncertainty matrices in Eq. (7). For a system without uncertainty, i.e., $\alpha = 0$ and $\beta = 0$, this augmented Riccati equation in Eq. (8) reduces to an ordinary Riccati equation that arises in the linear quadratic regulator problem.[10] ■

Corollary 1. Consider the matched uncertain linear system in Eq. (6) with the norm-bounded uncertainty matrices in Eq. (7). Let the parameter ϵ and the weighting matrix Q be given as in Theorem 1. Let $h > 0$ be a positive scalar and P the symmetric positive-definite solution of

$$(A + hI)^T P + P(A + hI) - (1 - \beta - \epsilon\alpha)PBB^T P + (\alpha/\epsilon)I + Q = 0 \tag{9}$$

Then, a robust stabilizing control law is given by $\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{x}(t)$, where $K = -\gamma B^T P$ with $\infty > \gamma \geq ½$, and the closed-loop system matrix $A_c(t) = A + BG(t) + [B + BH(t)]K$ is asymptotically stable with a prescribed degree of stability[10] h for all admissible uncertainty matrices $G(t)$ and $H(t)$ in Eq. (7). ■

Now we consider the matched uncertain linear system in Eq. (6) with uncertainty matrices $G(t) \in \mathbb{R}^{m\times n}$ and $H(t) \in \mathbb{R}^{m\times m}$ described by

$$G(t) = [g_{ij}(t)] = \sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(t) d_i e_j^T \tag{10a}$$

and

$$H(t) = [h_{ij}(t)] = \sum_{i=1}^{m} \sum_{j=1}^{m} h_{ij}(t) d_i d_j^T \tag{10b}$$

where $g_{ij}(t)$ and $h_{ij}(t)$ are the (i,j) entries of $G(t)$ and $H(t)$, respectively; $\boldsymbol{d}_i$ is defined as an $m \times 1$ unit vector with its ith element equal to 1, and equal to 0 otherwise; and $\boldsymbol{e}_i$, an $n \times 1$ unit vector, is similarly defined. The entries of the uncertainty matrices are bounded by

$$|g_{ij}(t)| \leq \bar{g}_{ij} \quad \text{for} \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{10c}$$

and

$$|h_{ij}(t)| \leq \bar{h}_{ij} \quad \text{for} \quad i = 1, \ldots, m, \quad j = 1, \ldots, m \tag{10d}$$

for all t.

To derive robust stabilizing controllers for the matched system in Eq. (6) with uncertainty described by Eqs. (10), we define the symmetric positive-semidefinite matrices $T \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{n\times n}$, and $V \in \mathbb{R}^{m\times m}$ as follows:

$$T \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{g}_{ij} d_i d_i^T = \text{diag}[t_1, \ldots, t_m] \quad t_i = \sum_{j=1}^{n} \bar{g}_{ij} \tag{11a}$$

$$U \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{g}_{ij} e_j e_j^T = \text{diag}[u_1, \ldots, u_n] \quad u_j = \sum_{i=1}^{m} \bar{g}_{ij} \tag{11b}$$

and

$$V \triangleq \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \bar{h}_{ij} (d_i d_i^T + d_j d_j^T) = \text{diag}[v_1, \ldots, v_m]$$

$$v_i = \frac{1}{2} \sum_{j=1}^{m} (\bar{h}_{ij} + \bar{h}_{ji}) \tag{11c}$$

From matching conditions in Eq. (2), we have $\|H(t)\| < 1$, which implies $I + ½[H(t) + H^T(t)] > 0$. As a result, we assume

$$I - V > 0 \tag{11d}$$

The following theorem guarantees that a robust stabilizing controller exists for the matched uncertain system in Eq. (6) with entry-bounded uncertainty matrices in Eqs. (10).

Theorem 2. Consider the matched uncertain linear system in Eq. (6) with entry-bounded uncertainty matrices described by Eqs. (10). Let $Q \in \mathbb{R}^{n\times n}$ be any given symmetric positive-definite weighting matrix and ϵ any selected positive scalar satisfying $\epsilon \in \{0, [1 - \max(v_i)]/\max(t_i)\}$. And let $P \in \mathbb{R}^{n\times n}$ be the symmetric positive-definite solution of the following Riccati equation:

$$A^T P + PA - PB(I - V - \epsilon T)B^T P + (1/\epsilon)U + Q = 0 \tag{12}$$

where T, U, and V are as defined in Eqs. (11). Then, a robust stabilizing control law is given by $\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{x}(t)$, where $K = -\gamma B^T P$ with $\infty > \gamma \geq ½$, and the closed-loop system matrix $A_c(t) = A + BG(t) + [B + BH(t)]K$ is asymptotically stable for all admissible uncertainty matrices $G(t)$ and $H(t)$ in Eqs. (10).

Proof. Define $Q(t)$ as in Theorem 1. From Eq. (12), it follows that

$$Q(t) = PB\{(2\gamma - 1)I + V + \gamma[H^T(t) + H(t)]\}B^T P + [\epsilon\, PBTB^T P + (1/\epsilon)U - G^T(t)B^T P - PBG(t)] + Q$$

Since

$$2V + H^T(t) + H(t) = \sum_{i=1}^{m} \sum_{j=1}^{m} [\bar{h}_{ij}(d_i d_i^T + d_j d_j^T) + h_{ij}(t)(d_i d_j^T + d_j d_i^T)]$$

$$\geq \sum_{i=1}^{m} \sum_{j=1}^{m} |h_{ij}(t)| [d_i \pm d_j]\,[d_i \pm d_j]^T \geq 0$$

and

$$[\epsilon PBTB^T P + (1/\epsilon)U - G^T(t)B^T P - PBG(t)]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} [\bar{g}_{ij}(\epsilon PBd_i d_i^T B^T P + (1/\epsilon)\, e_j e_j^T) - g_{ij}(t)\,(e_j d_i^T B^T P + PBd_i e_j^T)]$$

$$\geq \sum_{i=1}^{m} \sum_{j=1}^{n} |g_{ij}(t)| \left[\sqrt{\epsilon} PBd_i \pm \left(1/\sqrt{\epsilon}\right) e_j\right] \times [\sqrt{\epsilon} PBd_i \pm (1/\sqrt{\epsilon})e_j]^T \geq 0$$

It follows that

$$Q(t) \geq PB\left[(2\gamma - 1)I + V - 2\gamma V\right]B^T P + Q$$

$$= (2\gamma - 1)PB(I - V)B^T P + Q$$

Hence, $Q(t) \geq Q > 0$ for $\infty > \gamma \geq ½$. Thus, based on the Lyapunov stability theorem,[10] $A_c(t)$ is asymptotically stable for $\infty > \gamma \geq ½$. ■

Remark 3. When $T = \alpha I$, $U = \alpha I$, and $V = \beta I$, the Riccati expression in Eq. (12) reduces to the Riccati expression in Eq. (8) and the robust stabilizing controller obtained from Theorem 2 reduces to that of Theorem 1. In general, the robust stabilizing controllers obtained using Theorem 1 are more conservative than those obtained using Theorem 2. ■

Conclusion

Based on linear quadraric regulator theory and Lyapunov stability theory, new robust stabilizing control laws have been developed for stabilization of matched uncertain linear systems. It has been shown that a class of dynamic systems described by second-order monic vector differential equations often satisfies the matching conditions and that the robust stabilizing controllers always exist for the matched uncertain linear systems that contain either norm- or entry-bounded uncertainty matrices. The proposed robust stabilizing control law can be determined easily from the symmetric positive-definite solution of the augmented Riccati equation. In addition, the proposed approach is flexible in the sense that some adjustable parameters (such as ϵ, γ, and h, etc.) have been introduced in the derivations to achieve the stabilization of matched uncertain linear systems. Moreover, the proposed method can be applied to unstable and/or nonminimum phase-matched uncertain linear multivariable systems.

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Analysis of a Rotationally Accelerated Beam with Finite Tip Mass and Hub

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Nomenclature

A = cross-sectional area of the beam
a = radius of the hub
b = beam thickness
c = half the length of the tip mass
d = tip mass diameter
E = Young's modulus of elasticity
h = beam height
I = area moment of inertia of beam cross section
I_b = mass moment of inertia of the beam about $x = 0$
I_c = mass moment of inertia of the extended tip mass about its center of gravity
I_h = mass moment of inertia of the hub about center
I_t = mass moment of inertia of tip mass about $x = L$
L = length of the beam
L_h = hub length
L_L = beam length
L_t = tip mass length
M_b = beam mass
M_t = tip mass
P = period
r = radius of tip mass
$T(t)$ = torque applied to hub
T_o = torque magnitude
t = time
$v(x,t)$ = transverse deflection of the beam
x = coordinate along the undeflected beam
θ = hub angle of rotation
ρ = mass density of the beam
ρ_h = hub density
ρ_t = tip mass density
$\tilde{\phi}(x)$ = initial displacement distribution of the beam
$\tilde{\psi}(x)$ = initial velocity distribution of the beam

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Introduction

THE influence of element flexibility on the motion of a system is the subject of considerable current research.[1-13] In this Note, a rotationally accelerated Bernoulli beam is studied. Analytical expressions for beam tip displacement, hub rotation angle, and beam flexure are obtained. A parameter analysis of natural frequencies is presented.

Mathematical Model

The system shown in Fig. 1 consists of a slender flexible horizontal beam with a rectangular cross section that is attached to a rotating cylindrical rigid body supported in a cantilever fashion. A rigid-body tip is attached to the free end of the beam. The beam rotates in a horizontal plane generating in-plane bending. The nondimensional form of the governing equations neglecting terms in the square of the angular velocity are

Field equation:

$$\ddot{\eta} + \eta'''' + (\xi + \delta)\ddot{\theta} = 0, \qquad \xi \in (0,1) \tag{1}$$

Boundary conditions:

$$\eta(0,\tau) = 0, \qquad \eta'(0,\tau) = 0$$

$$\eta''(1,\tau) = -(\lambda/\sigma)[\ddot{\eta}(1,\tau) + (\delta + 1)\ddot{\theta}] - \zeta[\ddot{\eta}''(1,\tau) + \ddot{\theta}]$$

$$\eta'''(1,\tau) = (\lambda/\sigma)[\ddot{\eta}''(1,\tau) + \ddot{\theta}] + (1/\sigma)[\ddot{\eta}(1,\tau) + (\delta + 1)\ddot{\theta}] \tag{2}$$

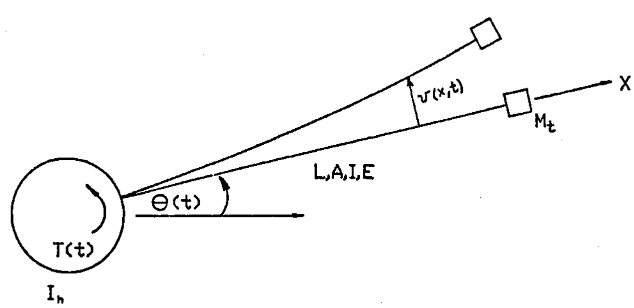

Fig. 1 Model problem.